Grad, Div & Curl

Grad

Narrator: (Francesca Hunt)
Skiers often zig-zag across a slope rather than going straight down it; they have to take a path that's not too steep, so they can control their speed. Even expert skiers like Carl can't ski down slopes of more than 25 to 30 !

Carl:
Going straight down the slope is the quickest way of getting down the slope. Alternatively you can go traverse which means skiing across the slope, that way you control your descent and you get down at a more controlled speed and more safely.

Narrator:
But what do we mean by 'the steepness of a path' on a hillside? How can we describe the steepness of the path that goes straight up or down the slope? And how is it related to the steepness of any other chosen path? To answer those questions we'll need to build a model of a hillside. We'll model the surface of the hillside as a function of two variables.
The plane you can see at the base is the x-y plane. The height z of any point on the surface itself is a unique scalar value, f(x,y). So the height z is a scalar field. In fact here f(x,y) is equal to sin x cos y, but that's not important. To find the slope of the steepest path at any point we'll use the tangent plane. The direction of the line of steepest upward slope on the tangent plane is shown by the arrow. No matter where we are on the surface, the arrow is always at right angles to the contour line passing through that point. So, if we look at the projection on the x - y plane, at any point we can define a vector, in the x-y plane, called grad f. The direction of grad f specifies the direction of the steepest slope while the magnitude of grad f is simply the magnitude of the steepest slope. You'll see that grad f is given by this definition involving the partial derivatives of f. The gradient vector field for the whole surface looks like this. However, as we've heard, skiers don't usually take the steepest path down a hillside. So how can we find the slope of a path in any other direction? It turns out that provided we know grad, it's not too hard to find. Let's go back and look at the hillside again. This is Brent Knoll in Somerset. In fact to find the slope of any path on this hillside we don't need a mathematical model; we can estimate the slope from a map! Each of the contours on this map represents a constant height above sea level. Earlier we saw that the magnitude and direction of the steepest slope of a path at any point are given by the gradient vector grad f. So let's find the gradient vector at this point A. We're assuming that the ground between these two contour lines is in a plane. Now we know that the direction of grad f is perpendicular to the contour line through A, and so will be parallel to a unit vector n uphill. The labelling on the contours shows that the rise in height along a path from the point A to the point B along the direction of grad is 10 metres. And as it's a map, that rise in height corresponds to going a horizontal distance AB. Using the scale on this map that works out to be 30 metres. So grad f is 'rise over run'; 10, divided by 30, in the direction n. In other words grad f is 1/3 n. So now we know the magnitude and direction of the gradient vector. But what's the slope of any other path? Say this one, passing through the points A and C? The direction of the path is parallel to this unit vector, u. The slope of the path through AC is given by the same rise of 10 metres marked on the contour lines but divided this time by the horizontal distance corresponding to AC; that's 50 metres. So the slope of AC is 1/5. But all we've done so far is find the slope from the map. How does all this relate to the gradient vector grad? Well, remember the contour lines are parallel, and so ABC is a right angled triangle. So AB is equal to AC times cos - which here means that 30 = 50 cos . Also u is a unit vector in the direction of AC, so cos is the dot product of the two unit vectors u and n. Putting all this together, we get the result that the slope of the path AC is equal to u. grad f. This is a general result, showing how the slope of any path is related to grad f. In fact you've

already seen another example of this result; the dot product of perpendicular vectors is zero, so the gradient vector must be perpendicular to the contour line at any point. So if skiers knew f, and the direction of any proposed path, they could work out the slope of the path by taking the dot product of the unit vector in the direction of their proposed path with grad f!