Grad, Div & Curl

Div

Narrator:

But the gradient vector isn't just useful for finding slopes of paths on hillsides. In a conducting material heat energy will flow from a hot region to a cooler one. The temperature field is a scalar field, while the flow of heat energy, having both magnitude and direction, must be a vector field. It turns out that they too are related. Engineers often have to calculate temperature distributions and heat flow patterns. The nuclear reactor in this power station generates heat by the controlled nuclear fission of uranium, heat which is then used to produce steam, which in turn drives turbines which generate electricity. Inside the reactor itself there are fuel elements consisting of long stainless steel rods containing uranium fuel. The rods get very hot as heat is generated inside them by nuclear fission.

Dr. Paul Hutt: (Nuclear Electric, Gloucester)

The temperature in the fuel can reach up to typically 900 degrees C in the centre of the fuel, that's conducted away to the edge of the fuel and out to the coolant. The fuel temperature on the edge of the fuel is typically about 450 degrees C. This is a fuel can for an AGL fuel rod. This is an empty fuel can, there's no uranium in this. It has a diameter of about one centimetre and is approximately a meter long. The helical ribs are to aid transfer of heat away from the fuel pins.

Narrator:

For reasons of thermodynamic efficiency it is essential to run the reactor as hot as possible, but on the other hand it's important not to allow the temperature to approach the melting points of any of the materials. How could we calculate the temperature distribution in the rod? To answer this question, we'll need first to identify the relevant scalar and vector fields in the rod, and then the relationships between them. The fuel rods inside most modern nuclear reactors are quite complicated in structure so let's look at a simple model, consisting of a single uranium rod. We'll assume that the rod is very long and cylindrical, with its axis along the z axis. Inside this single rod we'll assume that heat is being generated by nuclear fission at a constant rate uniformly throughout the rod that is the heat generating is continuous. And we'll also assume that the rod is in its steady state; that is, the temperature varies with position, but is constant in time. The rod will be very hot, but much hotter in the centre than on the outside surface where it's in contact with the coolant. The temperature of each point within the rod can be described by the scalar field we'll call capital . We're assuming the rod is very long and the heat flow is radial. So capital is a function only of x and y. There's also the field describing the flow of heat energy within the rod. This is a vector field, J. The magnitude of J is the rate of flow of heat energy across unit area, measured in Watts per square metre. So are there any other fields we should model? Well the heat energy is generated by fission reactions within the rod, so there must be another field which represents the heat energy source, that is, the rate of heat energy generation per unit volume at each point of the fuel rod. This is another scalar field, which we'll call S, and which will be measured in Watts per cubic metre. These three fields, the vector field J, and the two scalar fields, and S must be related in some way. But how? You might not be surprised to know that the flow of heat energy field J is related to the gradient of the temperature field: J = -grad . Using the definition of grad in terms of partial derivatives from earlier, that means J is related to by this equation. This equation shows that the flow of heat energy is directed against the temperature gradient, i.e. from hot to cold. But where does the heat source field S fit into all this? Well, S is the source of the heat energy produced while J describes how it flows, so we ought to be able to find a relationship between S and J by energy conservation. And you'll see later in the unit how this is done, resulting in this equation. The sum of these two partial derivatives of J is a scalar field, which we call the divergence of J, or div J for short. So we have the relationship: div J = S. Now we have two equations relating the fields , J and S in the fuel rod. However we could relate the temperature field and the heat source field S directly, by a single second order partial differential equation. I'd like you to have a go at that in a moment. You'll probably find it easier to use these representations of grad and div. See if you can eliminate J from the two equations to get a second order partial differential equation relating and S. This is the equation you should have got. It means that if we know the heat source field S, the thermal conductivity , and the boundary conditions, in this case that's the constant steady state temperature on the surface of the rod, we can solve this equation for the temperature field . This equation is called Poisson's equations, and it's the basis of calculations of temperature distributions in steady state heat flow problems.

Dr. Paul Hutt:
The practical use of this calculation for us is that we need to find essentially by calculation what the fuel temperatures are. We need to assure ourselves that the temperatures remain within acceptable limits, and working backwards we can then control the power which is generated in that fuel rod to keep the fuel within acceptable limits. Another part of the calculation we have to do is to determine the neutron distribution in the reactor and again that's a very similar equation – it's a diffusion equation for the neutron density. So again its a divergence of the…of the gradient of the neutron flux concentration, and we solve that over the whole reactor mathematically a very similar problem we have to solve finding the neutron density within a reactor.