Grad, Div & Curl

Curl

Narrator:

Both grad and div involve finding fields using partial derivatives. We'll look at yet another useful field. Once again it involves partial derivatives. Water can flow in many different and often complex ways. Let's look at a relatively simple case. What's the velocity field that describes the flow on the surface of a river? We can explore this field by watching floating objects. What kind of motion can you see here? There appear to be two types of motion. Downstream and rotation about a vertical axis. In fact if we're travelling along with the disc all we see is the rotation. This rotation is related to the velocity field. But how? To answer that question, we need to define the velocity field on the surface of the river. We'll define the surface velocity vector at any fixed point on the river, fixed, that is, relative to the river bank, as the velocity of any floating object as it passes through that point. To model the surface velocity we'll need to make some assumptions, both about the shape of the river and about the way that the water flows. We'll assume that there are no rocks affecting the flow. Let's also assume that the river is straight and of uniform width. And the flow is steady and streamline; it all flows downstream. So this is our model river.

To model the velocity field we first need some axes. And a function v(x,y) that describes the unique velocity vector at each point (x,y) on the surface. We'll assume that for this short stretch of the river v(x,y) doesn't depend on y. In other words the velocity at any point only depends on its distance x from the left bank. This implies a model surface velocity field of the form v(x,y) = u(x) j. But what sort of function is u(x)? Let's go back to the real river for a moment. The velocity of the water varies as we go across the river. And we can assume that at each of the banks the surface speed is zero. That is u(0) and u(d) are both zero - where d is the width of the river. Somewhere in between the surface speed will be a maximum. Let's assume that there's some symmetry and so that the maximum flow is midstream. What does this imply for u(x)? Well, the simplest possibility is parabolic; u(x) = Cx(d -x), where C is a constant. And so our model for the surface velocity field v is this. But how can a velocity field cause rotation? Let's go back to the model. Can you see what's happening? Let's subtract the downstream velocity of the centre of the disc. While the outer edge is being tugged downstream by the flow, the nearer edge is being tugged upstream. So the disc rotates. This indicates the presence of another vector field, one that describes the magnitude and direction of the rotation at any point in the river. Here the rotation is anticlockwise. So using the right hand rule the direction is vertically upwards, that is in the direction of k, the Cartesian unit vector in the z direction. This new vector field is called the curl of v, or just curl v. How does this vector curl v vary over the river? As we've seen, here the rotation is anticlockwise. But here it's clockwise. On the left-hand side of the river the velocity difference across the disc causes rotation in the positive k direction. While on the right hand side of the river the velocity difference causes rotation in the negative k direction. So what's the rotation midstream? Well midstream there's no rotation; in other words curl v = 0. The vector field showing curl v for the whole river looks like this. Everywhere curl v is perpendicular to the velocity field v. Any vector field can have a curl field associated with it. And later in this unit you'll see how curl v can be found from partial derivatives. For our model river the rotation was localised - there was rotation about each point. And it's local rotation that curl describes. Not the bulk rotation of water you see in river bends, or in swirling water. So what do you think is happening here? There's plenty of water movement - bulk rotation. But what about any local rotation? What would curl be? Let's model the problem by looking at the two-dimensional surface velocity. All the water's going round and round in a circle. But whilst that nearer the centre is flowing in a vortex, further out it gets slower and slower. So there are two different kinds of flow to model. Let's start with the water inside the vortex.

At the centre the velocity is zero. It's like a hurricane or a tornado, where the wind velocity is zero at its centre, its eye. As we move further out, the speed increases. If you ignore the bulk rotation causing it to flow around in a circular path you can see that the disc is also rotating about its centre. So with this velocity field there is local rotation. But what about the flow outside the vortex? Clearly there's still bulk rotation of the water. But what is the local rotation? This time the disc isn't rotating at all about its own centre so there's no local rotation. In other words this velocity field has a curl of zero. But why should anybody want to know about the curl of a vector field?

Professor Eric Priest: (St. Andrews University, Scotland)
In weather patterns you often find concentrations of the vorticity where the curl of the velocity is extremely large. These are called cyclones. A cyclone is a region where the pressure is lower than normal and so naturally the air tends to flow in towards the centre of the cyclone. But as it flows in, it doesn't just come in radially, it acquires a twist. In cyclones you get a build up of the vorticity and when the vorticity gets extremely intense, of course you get hurricanes and tornadoes and these can do a great deal of damage. So it's very important that we understand how the vorticity is built up and how tornadoes and hurricanes behave so that we can predict them better in future. I'm particularly interested in the sun. There are very strong magnetic fields and the curl of the magnetic field is in fact the electric current. In most of the sun the current is zero, there's no curl. But in small regions where the curl is very large, you find dynamic phenomena produced. As you go away from the surface of the sun you might expect going away from a hot body that the temperature would get lower. In fact, the opposite occurs, it actually increases, and in the atmosphere of the sun the temperature is several million degrees, five million degrees, really hot. There are giant tubes of magnetic flux in the atmosphere. When these have no curl they just sit there in a quiet state for months at a time. But when the curl builds up, when the electric current builds up, eventually they could reach a stage where they go unstable and they erupt outwards from the sun and produce enormous ejections of mass. Knowing about the curl enables us to understand how and why these ejections occur.