Maths as others see it

Looking for Theta

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Now let's consider an example between points that are not on the equator. The Breitling 3 balloon started its round-the-world flight from Château-d'Oex, in Switzerland, at 46 point five degrees north and seven point one degrees east, and its first leg ended at the first waypoint in northern Saudi Arabia, at 19 degrees north and 42 point five degrees east. So, how can you calculate the distance between the start and the first waypoint? Again, you need to calculate the angle at the centre of the earth subtended by the arc of the great circle that connects the two points. But now you're dealing with a great circle that's neither a line of longitude nor the equator – so where do you begin?

The idea is to find theta, the angle subtending the great arc, by constructing some triangles and using trigonometry. The important thing here is to understand the principles involved, so don't be too concerned if you miss some of the detail. Let's label the start S, the waypoint W, and the centre of the earth O. The two lines of longitude meet at the North Pole, N. Now draw the radius from O to N, and the tangent at N to the longitude of S. And then extend the radius O S to meet this tangent at S dash. Similarly, draw the tangent at N to the longitude of W and extend the line O W to meet this tangent at W dash. Join S dash to W dash. Now remember we want to find theta, which is in triangle O S dash W dash. If we knew the sides of this triangle, we could use the cosine rule to find theta. So we want to find the lengths of the three sides – O S dash, O W dash and S dash W dash. Now, O S dash is also in triangle O N S dash, so we might be able to find the length of O S dash from this triangle. O W dash is also in triangle O N W dash, so we might be able to find that from this triangle. And S dash W dash is also in triangle N S dash W dash, so we might be able to find that from this triangle.

Let's look at each of these triangles in turn. First, look at triangle O N S dash, which is a right-angled triangle. Now, O N is the radius of the earth. So we know this. The latitude of S is 46 point five degrees, which is the same as the angle alpha in the triangle, because N S dash is parallel to the equatorial radius. Since O N S dash is a right-angled triangle and we know one side, O N, and one angle, we can use the trigonometric ratios to find the other two sides: O S dash and N S dash. The sine of alpha is the side opposite alpha over the hypotenuse, that is O N over O S dash. Now, to get O S dash on its own, multiply both sides by O S dash and then divide by sine alpha. Putting in the numbers means that O S dash is 8783 kilometres. So now we know O S dash. We can also find the third side in the triangle, N S dash, which will prove useful later on. The cosine of alpha is the side adjacent to it, that's N S dash, over the hypotenuse, O S dash. To get N S dash on its own, multiply both sides by O S dash, and put in the numbers. So now we know N S dash. Similar reasoning can be applied to triangle O N W dash. The latitude of W is 19 degrees, which is the same as the angle beta in the triangle. The sine of beta is the side opposite beta, O N, over the hypotenuse, O W dash. To get O W dash, multiply both sides by O W dash and divide by sine beta. Putting in the numbers gives O W dash as 19569 kilometres. So, now we know O W dash. We can also find the third side in the triangle, N W dash, which will prove useful later on. The cosine of beta is the side adjacent to it, N W dash, over the hypotenuse, O W dash. Multiply both sides by O W dash to get N W dash, and put in the numbers, getting N W dash as 18503 kilometres. Now we know N W dash. So, let's see where we are in our strategy. What we want is theta. And to find that, we wanted O S dash, O W dash and S dash W dash. Well, now we know O S dash and O W dash. But we still need to find S dash W dash, which is in triangle N S dash W dash. Unfortunately, this is not a right-angled triangle, but we do know two sides: N S dash and N W dash.

We can also find the angle gamma from the angles of longitude of S and W. It’s the difference between the longitude of S and that of W, that is 42 point five degrees minus seven point one degrees, which is 35 point four degrees.
So, we know two sides and the included angle in triangle N S dash W dash. So we can use the cosine formula, in this form, to find the third side that we want.
In this case: c is S dash W dash, because it is the side opposite the angle; a is N S dash; and b is N W dash. Putting in the numbers and taking the square root, S dash W dash is 14 019 kilometres.
So, now we have the S dash W dash that we wanted.
The final stage is to find theta, using the cosine formula in the other form, for triangle O S dash W dash.
In this case: c is S dash W dash; a is O S dash; and b is O W dash. Putting in the numbers and using the inverse cosine function gives theta as just under 40 degrees.
In order to use the formula for the arc length – l equals R times the angle theta – we need to convert theta into radians, by multiplying by pi over 180. This gives the great-circle distance from the start to the first waypoint as 4437 kilometres (to the nearest kilometre).